

# STABILIZER GROUP OF GENERALIZED DETERMINANT

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**ABSTRACT.** In this paper, we introduce the notion of generalized determinant and determine the stabilizer group in  $GL(\text{Mat}_n(K))$  of the generalized determinant.

## 1. INTRODUCTION.

For an  $n \times n$  matrix  $A := (a_{ij})_{1 \leq i, j \leq n}$ , we will define the *even determinant*  $\det_{A_n}(A) := \sum_{\sigma \in A_n} \prod_{i=1}^n a_{i\sigma(i)}$  and the *odd determinant*  $\det_{\bar{A}_n}(A) := \sum_{\sigma \in \bar{A}_n} \prod_{i=1}^n a_{i\sigma(i)}$  where  $\bar{A}_n$  is the set  $S_n \setminus A_n$ . Let  $K$  be a field of characteristic 0. Let  $\alpha, \beta \in K$ . We introduce the *generalized determinant*  $\det^{(\alpha, \beta)}(A)$  of an  $n \times n$  matrix  $A$  by

$$\det^{(\alpha, \beta)}(A) := \alpha \det_{A_n}(A) + \beta \det_{\bar{A}_n}(A).$$

Let  $X := (x_{ij})_{1 \leq i, j \leq n}$  where  $x_{ij}$  are the standard basis of  $\text{Mat}_n^*$ . Write  $\det_n := \det(X)$ ,  $\text{perm}_n := \text{perm}(X)$  and  $\det_n^{(\alpha, \beta)} := \det^{(\alpha, \beta)}(X)$ . The stabilizer group  $\text{Stab}(f)$  of  $f \in \text{Sym}^n(\text{Mat}_n^*)$  is defined as  $\text{Stab}(f) := \{T \in GL(\text{Mat}_n(K)) \mid T \cdot f = f\}$ . If  $\alpha = -\beta \neq 0$ , then  $\text{Stab}(\det^{(\alpha, -\alpha)}) = \text{Stab}(\det_n)$ , which had been determined by Frobenius [1] as follows.

**Theorem 1** (Frobenius). *It holds that  $\text{Stab}(\det_n) = \{X \mapsto PXQ \text{ or } P^tXQ \mid \det P \det Q = 1\}$ . Here  $P, Q \in GL_n(K)$ .*

On the other hand, if  $\alpha = \beta \neq 0$ , then  $\text{Stab}(\det^{(\alpha, \alpha)}) = \text{Stab}(\text{perm}_n)$ , which was determined by Marcus and May [2] as follows.

**Theorem 2** (Marcus and May). *Let  $n \geq 3$ . It holds that  $\text{Stab}(\text{perm}_n) = \{X \mapsto LPXQR \text{ or } LP^tXQR \mid \det L \det R = 1\}$  where  $P$  and  $Q$  are permutation matrices,  $L$  and  $R$  are diagonal matrices.*

Our purpose is to relate the two results by  $\mathbb{P}^1(K)$ -family  $\{\det_n^{(\alpha, \beta)} \mid [\alpha : \beta] \in \mathbb{P}^1(K)\}$ . Our main result is the following.

**Theorem 3.** *Let  $n \geq 5$ . If  $\alpha \neq \pm\beta$ , then  $\text{Stab}(\det_n^{(\alpha, \beta)}) = \text{Stab}(\det_n) \cap \text{Stab}(\text{perm}_n)$ .*

## 2. NOTATIONS.

We define a submatrix  $X_{l_1 \dots l_r}^{k_1 \dots k_r}$  of  $X$  by  $X_{l_1 \dots l_r}^{k_1 \dots k_r} := (x_{k_i l_j})_{1 \leq i, j \leq r}$ . Let  $P_r^{(\alpha, \beta)}(X) := (\det^{(\alpha, \beta)}(X_{l_1 \dots l_r}^{k_1 \dots k_r}))_{\substack{1 \leq k_1 < \dots < k_r \leq n \\ 1 \leq l_1 < \dots < l_r \leq n}}$  be a  $\binom{n}{r} \times \binom{n}{r}$  matrix. Specially, we write  $P_r(X) := P_r^{(1, 0)}(X)$  and  $\bar{P}_r(X) := P_r^{(0, 1)}(X)$ .

**Lemma 4.** *Let  $n \geq 4$ . If  $\det^{(\alpha, \beta)}(T(X)) = \det^{(\alpha, \beta)}(X)$ , then  $\det^{(\alpha, \beta)}(T(X))_{l_1 l_2}^{k_1 k_2}$  and  $\det^{(\beta, \alpha)}(T(X))_{l_1 l_2}^{k_1 k_2}$  are expressible as a linear combination of  $\det^{(\alpha, \beta)}(X_{l'_1 l'_2}^{k'_1 k'_2})$  and  $\det^{(\beta, \alpha)}(X_{l'_1 l'_2}^{k'_1 k'_2})$  ( $1 \leq k'_1, k'_2, l'_1, l'_2 \leq n$ ) respectively.*

*Proof.* Let  $Y := T(X)$ . We can write each entry of  $X = T^{-1}(Y)$  as  $x_{st} = \sum_{p,q=1}^n g_{st}^{pq} y_{pq}$ . We first prove the case  $\det^{(\alpha,\beta)}(T(X)_{l_1 l_2}^{k_1 k_2})$ . Since  $n \geq 4$ , there exists  $\sigma \in A_n$  such that  $\sigma(k_1) = l_1$ ,  $\sigma(k_2) = l_2$ . The permutation  $(l_1 \ l_2)\sigma \in \bar{A}_n$  satisfies  $(l_1 \ l_2)\sigma(k_1) = l_2$  and  $(l_1 \ l_2)\sigma(k_2) = l_1$ . We compute

$$\begin{aligned} \det^{(\alpha,\beta)}(Y_{l_1 l_2}^{k_1 k_2}) &= \det^{(\alpha,\beta)} \begin{pmatrix} y_{k_1 l_1} & y_{k_1 l_2} \\ y_{k_2 l_1} & y_{k_2 l_2} \end{pmatrix} \\ &= \frac{\partial^{n-2}}{\partial y_{1\sigma(1)} \cdots \partial y_{k_1\sigma(k_1)} \cdots \partial y_{k_2\sigma(k_2)} \cdots \partial y_{n\sigma(n)}} \det^{(\alpha,\beta)}(Y) \\ &= \frac{\partial^{n-2}}{\partial y_{1\sigma(1)} \cdots \partial y_{k_1\sigma(k_1)} \cdots \partial y_{k_2\sigma(k_2)} \cdots \partial y_{n\sigma(n)}} \det^{(\alpha,\beta)}(X). \end{aligned}$$

To compute it, we use

$$\frac{\partial}{\partial y_{pq}} \det^{(\alpha,\beta)}(X_{l_1 \cdots l_r}^{k_1 \cdots k_r}) = \sum_{s,t=1}^n \frac{\partial x_{st}}{\partial y_{pq}} \frac{\partial}{\partial x_{st}} \det^{(\alpha,\beta)}(X_{l_1 \cdots l_r}^{k_1 \cdots k_r})$$

and

$$\frac{\partial}{\partial y_{pq}} \det^{(\beta,\alpha)}(X_{l_1 \cdots l_r}^{k_1 \cdots k_r}) = \sum_{s,t=1}^n \frac{\partial x_{st}}{\partial y_{pq}} \frac{\partial}{\partial x_{st}} \det^{(\beta,\alpha)}(X_{l_1 \cdots l_r}^{k_1 \cdots k_r}).$$

We have  $\frac{\partial x_{st}}{\partial y_{pq}} = g_{st}^{pq} \in K$  and  $\frac{\partial}{\partial x_{st}} \det^{(\alpha,\beta)}(X_{l_1 \cdots l_r}^{k_1 \cdots k_r})$  is equal to  $\det^{(\alpha,\beta)}(X_{l_1 \cdots \hat{l} \cdots l_r}^{k_1 \cdots \hat{s} \cdots k_r})$ ,  $\det^{(\beta,\alpha)}(X_{l_1 \cdots \hat{l} \cdots l_r}^{k_1 \cdots \hat{s} \cdots k_r})$  or 0. Hence, differentiating  $n-2$  times, the lemma follows.

We now turn to the case  $\det^{(\beta,\alpha)}(T(X)_{l_1 l_2}^{k_1 k_2})$ . Since  $n \geq 4$ , there exists  $\sigma \in A_n$  such that  $\sigma(k_1) = l_2$ ,  $\sigma(k_2) = l_1$ . The permutation  $(l_1 \ l_2)\sigma \in \bar{A}_n$  satisfies  $(l_1 \ l_2)\sigma(k_1) = l_1$ ,  $(l_1 \ l_2)\sigma(k_2) = l_2$ . Hence the same proof works for  $\det^{(\beta,\alpha)}(T(X)_{l_1 l_2}^{k_1 k_2})$ .  $\square$

**Lemma 5.** *Let  $A \in \text{Mat}_n(K)$ . If  $P_2(A) = \bar{P}_2(A) = 0$ , then  $A$  is 0, a row matrix or a column matrix.*

*Proof.* Consider  $A \neq 0$ . Without loss of generality we can assume  $a_{11} \neq 0$ . Then  $P_2(A) = 0$  implies  $a_{ij} = 0$  ( $2 \leq \forall i, j \leq n$ ). If there is  $j \geq 2$  such that  $a_{1j} \neq 0$ , then  $a_{i1} = 0$  ( $2 \leq \forall i \leq n$ ) from  $\bar{P}_2(A) = 0$ . Thus  $A$  is a row matrix. If there is  $i \geq 2$  such that  $a_{i1} \neq 0$ , then  $a_{1j} = 0$  ( $2 \leq \forall j \leq n$ ) from  $\bar{P}_2(A) = 0$ . Thus  $A$  is a column matrix.  $\square$

**Lemma 6.** *Define  $F_{ij} := T(E_{ij})$ . If  $\alpha \neq \pm\beta$ , then the number of non-zero entries in  $F_{ij}$  is one.*

*Proof.* By Lemma 4, both  $\det^{(\alpha,\beta)}(F_{ij l_1 l_2}^{k_1 k_2})$  and  $\det^{(\beta,\alpha)}(F_{ij l_1 l_2}^{k_1 k_2})$  are linear combination of  $\det^{(\alpha,\beta)}(E_{ij l'_1 l'_2}^{k'_1 k'_2})$  and  $\det^{(\beta,\alpha)}(E_{ij l'_1 l'_2}^{k'_1 k'_2})$  ( $1 \leq k'_1, k'_2, l'_1, l'_2 \leq n$ ) respectively. We thus get  $P_2^{(\alpha,\beta)}(F_{ij}) = \alpha P_2(F_{ij}) + \beta \bar{P}_2(F_{ij}) = 0$  and  $P_2^{(\beta,\alpha)}(F_{ij}) = \beta P_2(F_{ij}) + \alpha \bar{P}_2(F_{ij}) = 0$ . Since  $\alpha^2 - \beta^2 \neq 0$ ,  $P_2(F_{ij}) = \bar{P}_2(F_{ij}) = 0$ . Applying Lemma 5, we see that  $F_{ij}$  is a row matrix or a column matrix.

Suppose that the number of non-zero entries in  $F_{ij}$  is two or more. Let us assume that  $F_{ij}$  is a row matrix with non-zero entries in the  $i'$ 'th row. Since  $F_{ij} + F_{it} = T(E_{ij} + E_{it})$ ,  $F_{ij} + F_{tj} = T(E_{ij} + E_{tj})$ , we have  $P_2(F_{ij} + F_{it}) = \bar{P}_2(F_{ij} + F_{it}) = P_2(F_{ij} + F_{tj}) = \bar{P}_2(F_{ij} + F_{tj}) = 0$  ( $1 < \forall t \leq n$ ). By Lemma 5,  $F_{ij} + F_{it}$  and  $F_{ij} + F_{tj}$  are row matrices or column matrices, so that  $F_{it}$  and  $F_{tj}$  are row matrices lying in the  $i'$ 'th row. However

$\dim \text{span}\{E_{i1}, E_{i2}, \dots, E_{in}, E_{1j}, \dots, E_{nj}\} > \dim \text{span}\{E_{i'1}, E_{i'2}, \dots, E_{i'n}\}$ , which contradicts the fact that  $T$  is non-singular. The same proof works in the case that  $F_{ij}$  is a column matrix.  $\square$

By Lemma 6, we have  $T(E_{ij}) = c_{ij}E_{i'j'}$ . Since  $T$  is non-singular,  $c_{ij} \neq 0$  and  $(i, j) \neq (s, t)$  implies  $(i', j') \neq (s', t')$ . Hereafter, we always assume  $\alpha \neq \pm\beta$  and define maps  $\mu, \lambda$  by  $T(E_{ij}) = c_{ij}E_{\mu(i,j)\lambda(i,j)}$ .

**Lemma 7.** *There exist permutation matrices  $P := (\delta_{i\sigma(j)})_{1 \leq i, j \leq n}$ ,  $Q := (\delta_{i\tau(j)})_{1 \leq i, j \leq n}$  where  $\text{sgn}(\sigma)\text{sgn}(\tau) = 1$ , and a matrix  $C := (c_{ij})_{1 \leq i, j \leq n} \in M_{n,n}(K)$  with  $\forall c_{ij} \neq 0$  such that  $T(X) = C * PXQ$  or  $T(X) = C * P^tXQ$  (the operation  $*$  is the Hadamard product).*

*Proof.* We may assume that  $\mu(1, 1) = 1$  and  $\lambda(1, 1) = 1$  by swapping rows or columns even number of times, that is, the number of row and column transpositions are both even or both odd. Since  $\text{rank}(E_{11} + E_{22}) = 2$ , we have  $P_2(F_{11} + F_{22}) \neq 0$  or  $\overline{P}_2(F_{11} + F_{22}) \neq 0$ . It follows that  $\mu(2, 2) \geq 2$  and  $\lambda(2, 2) \geq 2$ . Therefore, swapping rows or columns even number of times properly, we may assume that  $\mu(2, 2) = 2$  and  $\lambda(2, 2) = 2$ . By continuing the same argument, we can assume  $\mu(i, i) = i$  ( $1 \leq \forall i \leq n$ ) and  $\lambda(i, i) = i$  ( $1 \leq \forall i \leq n-2$ ). There are two possibilities: (i)  $\lambda(n-1, n-1) = n-1$  and  $\lambda(n, n) = n$ , (ii)  $\lambda(n-1, n-1) = n$  and  $\lambda(n, n) = n-1$ . However, the case (ii) never happens because the coefficients of  $x_{11} \dots x_{n-1, n-1} x_{nn}$  in  $\det^{(\alpha, \beta)}(T(X))$  and in  $\det^{(\alpha, \beta)}(X)$  are different. Therefore, we conclude that

$$T(X) = C * P \begin{pmatrix} x_{11} & & & * \\ & \ddots & & \\ & & x_{n-1, n-1} & \\ * & & & x_{nn} \end{pmatrix} Q$$

where  $\text{sgn}(\sigma)\text{sgn}(\tau) = 1$ . To continue the argument, we may assume  $P = Q = I_n$  that is  $\mu(i, i) = i$  and  $\lambda(i, i) = i$  ( $1 \leq \forall i \leq n$ ) without loss of generality.

By  $P_2(E_{11} + E_{12}) = \overline{P}_2(E_{11} + E_{12}) = 0$ , we get  $\mu(1, 2) = 1$  or  $\lambda(1, 2) = 1$ . By  $P_2(E_{22} + E_{12}) = \overline{P}_2(E_{22} + E_{12}) = 0$ , we also get  $\mu(1, 2) = 2$  or  $\lambda(1, 2) = 2$ . Combining these, we obtain two possibilities: (I)  $\mu(1, 2) = 1$  and  $\lambda(1, 2) = 2$ , (II)  $\mu(1, 2) = 2$  and  $\lambda(1, 2) = 1$ .

Suppose first that (I) holds. Let  $3 \leq \gamma \leq n$ . By  $P_2(E_{11} + E_{1\gamma}) = \overline{P}_2(E_{11} + E_{1\gamma}) = 0$ , we get  $\mu(1, \gamma) = 1$  or  $\lambda(1, \gamma) = 1$ . By  $P_2(E_{12} + E_{1\gamma}) = \overline{P}_2(E_{12} + E_{1\gamma}) = 0$ , we also get  $\mu(1, \gamma) = 1$  or  $\lambda(1, \gamma) = 2$ . Combining these gives  $\mu(1, \gamma) = 1$ . By  $P_2(E_{\gamma\gamma} + E_{1\gamma}) = \overline{P}_2(E_{\gamma\gamma} + E_{1\gamma}) = 0$ , we obtain  $\lambda(1, \gamma) = \gamma$ .

Let  $\delta \neq 1$ . By  $P_2(E_{11} + E_{\delta 1}) = \overline{P}_2(E_{11} + E_{\delta 1}) = 0$ , we have  $\mu(\delta, 1) = 1$  or  $\lambda(\delta, 1) = 1$ . However  $\mu(1, \gamma) = 1$  ( $1 \leq \gamma \leq n$ ) gives  $\mu(\delta, 1) \neq 1$  as  $T$  is non-singular. Hence  $\lambda(\delta, 1) = 1$ . By  $P_2(E_{\delta\delta} + E_{\delta 1}) = \overline{P}_2(E_{\delta\delta} + E_{\delta 1}) = 0$ , we obtain  $\mu(\delta, 1) = \delta$ .

Let  $1 < \gamma \neq \delta \leq n$ . By  $P_2(E_{\delta 1} + E_{\delta\gamma}) = \overline{P}_2(E_{\delta 1} + E_{\delta\gamma}) = 0$ , we get  $\mu(\delta, \gamma) = \delta$  or  $\lambda(\delta, \gamma) = 1$  but the latter is impossible. By  $P_2(E_{1\gamma} + E_{\delta\gamma}) = \overline{P}_2(E_{1\gamma} + E_{\delta\gamma}) = 0$ , we also get  $\lambda(\delta, \gamma) = \gamma$ .

By the above argument, we obtain  $\mu(i, j) = i$  and  $\lambda(i, j) = j$  ( $1 \leq \forall i, j \leq n$ ) that is

$$T(X) = C * PXQ \quad (\text{sgn}(\sigma)\text{sgn}(\tau) = 1).$$

The same proof works for the case (II) and we also obtain

$$T(X) = C * P^tXQ \quad (\text{sgn}(\sigma)\text{sgn}(\tau) = 1),$$

if (II) holds. □

**Lemma 8.** *If  $n \geq 5$ , then  $\text{rank}(C) = 1$ .*

*Proof.* Comparing the coefficients of  $\det^{(\alpha, \beta)}(X)$  and  $\det^{(\alpha, \beta)}(C * PXQ)$ , we obtain

$$\begin{aligned} (1) \quad & c_{1w(1)} \cdots c_{nw(n)} = 1 \quad (\forall w \in A_n) \quad (\beta = 0) \\ (2) \quad & c_{1w(1)} \cdots c_{nw(n)} = 1 \quad (\forall w \in \bar{A}_n) \quad (\alpha = 0) \\ (3) \quad & c_{1w(1)} \cdots c_{nw(n)} = 1 \quad (\forall w \in S_n) \quad (\alpha \neq 0, \beta \neq 0). \end{aligned}$$

Let us solve these simultaneous equations.

For preparation, we first consider the case  $n = 4$ . Let  $\beta = 0$ . By  $c_{11}c_{22}(c_{33}c_{44}) = 1$ ,  $c_{12}c_{23}(c_{31}c_{44}) = 1$  and  $c_{13}c_{21}(c_{32}c_{44}) = 1$ , we can write

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31}c_{44} & c_{32}c_{44} & c_{33}c_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & u & v \\ \frac{1}{a_{12}v} & \frac{1}{a_{13}a_{21}} & \frac{1}{a_{11}u} \end{pmatrix}.$$

Set  $a_{31} := c_{31}$ . Then we have

$$c_{44} = \frac{1}{a_{12}a_{31}v}, \quad c_{32} = \frac{a_{12}a_{31}v}{a_{13}a_{21}}, \quad c_{33} = \frac{a_{12}a_{31}v}{a_{11}u}.$$

Set  $a_{14} := c_{14}$ . By  $c_{14}c_{23}c_{32}c_{41} = 1$ , we have

$$a_{14} \cdot v \cdot \frac{a_{12}a_{31}v}{a_{13}a_{21}} \cdot c_{41} = 1 \quad \therefore c_{41} = \frac{a_{13}a_{21}}{a_{12}a_{14}a_{31}v^2}.$$

By  $c_{12}c_{24}c_{33}c_{41} = 1$  and  $c_{13}c_{22}c_{34}c_{41} = 1$ , we have

$$a_{12} \cdot c_{24} \cdot \frac{a_{12}a_{31}v}{a_{11}u} \cdot \frac{a_{13}a_{21}}{a_{12}a_{14}a_{31}v^2} = 1 \quad \therefore c_{24} = \frac{a_{11}a_{14}uv}{a_{12}a_{13}a_{21}}$$

$$a_{13} \cdot u \cdot c_{34} \cdot \frac{a_{13}a_{21}}{a_{12}a_{14}a_{31}v^2} = 1 \quad \therefore c_{34} = \frac{a_{12}a_{14}a_{31}v^2}{a_{13}^2a_{21}u}.$$

By  $c_{13}c_{24}c_{31}c_{42} = 1$  and  $c_{14}c_{21}c_{33}c_{42} = 1$ , we have

$$a_{13} \cdot \frac{a_{11}a_{14}uv}{a_{12}a_{13}a_{21}} \cdot a_{31} \cdot c_{42} = 1 \quad \therefore c_{42} = \frac{a_{12}a_{21}}{a_{11}a_{14}a_{31}uv}$$

$$a_{14} \cdot a_{21} \cdot \frac{a_{12}a_{31}v}{a_{11}u} \cdot c_{42} = 1 \quad \therefore c_{42} = \frac{a_{11}u}{a_{12}a_{14}a_{21}a_{31}v}.$$

Combining these yields

$$u = \pm \frac{a_{12}a_{21}}{a_{11}}.$$

By  $c_{11}c_{24}c_{32}c_{43} = 1$  and  $c_{14}c_{22}c_{31}c_{43} = 1$ , we have

$$a_{11} \cdot \frac{a_{11}a_{14}uv}{a_{12}a_{13}a_{21}} \cdot \frac{a_{12}a_{31}v}{a_{13}a_{21}} \cdot c_{43} = 1 \quad \therefore c_{43} = \frac{a_{13}^2a_{21}^2}{a_{11}^2a_{14}a_{31}uv^2}$$

$$a_{14} \cdot u \cdot a_{31} \cdot c_{43} = 1 \quad \therefore c_{43} = \frac{1}{a_{14}a_{31}u}.$$

Combining these yields

$$v = \pm \frac{a_{13}a_{21}}{a_{11}}.$$

Now, set  $a_{41} := c_{41} = \frac{a_{11}^2}{a_{12}a_{13}a_{14}a_{21}a_{31}}$ . Summarizing the above, we obtain

$$(4) \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \varepsilon_u & \varepsilon_v & \varepsilon_u \varepsilon_v \\ 1 & \varepsilon_v & \varepsilon_u \varepsilon_v & \varepsilon_u \\ 1 & \varepsilon_u \varepsilon_v & \varepsilon_u & \varepsilon_v \end{pmatrix} * \left( \frac{a_{i1}a_{1j}}{a_{11}} \right)_{1 \leq i,j \leq 4}$$

where  $\varepsilon_u, \varepsilon_v \in \{+1, -1\}$  and  $a_{11} \cdots a_{14}a_{11} \cdots a_{41} = a_{11}^4$ . Conversely, the matrix C is a solution of the simultaneous equation (1).

Let  $\alpha = 0$ . Interchanging the 3rd and the 4th row of (4), we obtain

$$(5) \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \varepsilon_u & \varepsilon_v & \varepsilon_u \varepsilon_v \\ 1 & \varepsilon_u \varepsilon_v & \varepsilon_u & \varepsilon_v \\ 1 & \varepsilon_v & \varepsilon_u \varepsilon_v & \varepsilon_u \end{pmatrix} * \left( \frac{a_{i1}a_{1j}}{a_{11}} \right)_{1 \leq i,j \leq 4}$$

where  $\varepsilon_u, \varepsilon_v \in \{+1, -1\}$  and  $a_{11} \cdots a_{14}a_{11} \cdots a_{41} = a_{11}^4$  as a solution of the simultaneous equation (2).

Let  $\alpha \neq 0$  and  $\beta \neq 0$ . Combining (4) and (5), we obtain

$$(6) \quad C = \left( \frac{a_{i1}a_{1j}}{a_{11}} \right)_{1 \leq i,j \leq 4}$$

where  $a_{11} \cdots a_{14}a_{11} \cdots a_{41} = a_{11}^4$  as a solution of the simultaneous equation (3).

Let us consider the simultaneous equations for  $n \geq 5$ . We prove that the solution of the each simultaneous equations (1),(2),(3) is expressible as

$$(7) \quad C = \left( \frac{a_{i1}a_{1j}}{a_{11}} \right)_{1 \leq i,j \leq n}$$

where  $a_{11} \cdots a_{1n}a_{11} \cdots a_{n1} = a_{11}^n$  respectively by induction of the matrix size  $n$ .

In the case  $n = 5$  and  $\beta = 0$ , using the fact that any solution of (1) may be written as (4), we can write

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41}c_{55} & c_{42}c_{55} & c_{43}c_{55} & c_{44}c_{55} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \varepsilon_u \frac{a_{21}a_{12}}{a_{11}} & \varepsilon_v \frac{a_{21}a_{13}}{a_{11}} & \varepsilon_u \varepsilon_v \frac{a_{21}a_{14}}{a_{11}} \\ a_{31} & \varepsilon_v \frac{a_{31}a_{12}}{a_{11}} & \varepsilon_u \varepsilon_v \frac{a_{31}a_{13}}{a_{11}} & \varepsilon_u \frac{a_{31}a_{14}}{a_{11}} \\ z & \varepsilon_u \varepsilon_v \frac{za_{12}}{a_{11}} & \varepsilon_u \frac{za_{13}}{a_{11}} & \varepsilon_v \frac{za_{14}}{a_{11}} \end{pmatrix}$$

where  $\varepsilon_u, \varepsilon_v \in \{+1, -1\}$  and  $a_{11} \cdots a_{14}a_{11} \cdots a_{31}z = a_{11}^4$ . Set  $a_{41} := c_{41}$ . Then  $c_{55} = \frac{z}{a_{41}}$ ,  $c_{42} = \varepsilon_u \varepsilon_v \frac{a_{41}a_{12}}{a_{11}}$ ,  $c_{43} = \varepsilon_u \frac{a_{41}a_{13}}{a_{11}}$ ,  $c_{44} = \varepsilon_v \frac{a_{41}a_{14}}{a_{11}}$ . Set  $a_{15} := c_{15}$ . If  $\varepsilon_u = -1$  or  $\varepsilon_v = -1$ , there exist  $w \in A_n$  such that  $w(1) = 5$ ,  $w(5) = 1$ ,  $c_{iw(i)} = \frac{a_{i1}a_{1w(i)}}{a_{11}}$  ( $2 \leq i \leq 4$ ) and  $w' \in A_n$  such that  $w'(1) = 5$ ,  $w'(5) = 1$ ,  $c_{iw'(i)} = -\frac{a_{i1}a_{1w'(i)}}{a_{11}}$  ( $2 \leq i \leq 4$ ). Then  $c_{15}c_{2w(2)}c_{3w(3)}c_{4w(4)}c_{51} \neq c_{15}c_{2w'(2)}c_{3w'(3)}c_{4w'(4)}c_{51}$  and one of the two cannot be equal to 1, a contradiction. Thus  $\varepsilon_u = \varepsilon_v = +1$ . The similar consideration applies to the case  $n = 5$  and  $\alpha = 0$ .

Therefore, assuming (7) to hold for  $n - 1$ , we can write

$$\begin{pmatrix} c_{11} & \cdots & c_{1,n-1} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n-1} \\ c_{n-1,1}c_{nn} & \cdots & c_{n-1,n-1}c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1,n-1} \\ \vdots & \frac{a_{21}a_{12}}{a_{11}} & \cdots & \frac{a_{21}a_{1,n-1}}{a_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & \frac{a_{n-2,1}a_{12}}{a_{11}} & \cdots & \frac{a_{n-2,1}a_{1,n-1}}{a_{11}} \\ z & \frac{za_{12}}{a_{11}} & \cdots & \frac{za_{1,n-1}}{a_{11}} \end{pmatrix}$$

where  $a_{11} \cdots a_{1,n-1}a_{11} \cdots a_{n-2,1}z = a_{11}^{n-1}$ . Set  $a_{n-1,1} := c_{n-1,1}$ . Then we have

$$c_{nn} = \frac{z}{a_{n-1,1}}, \quad c_{n-1,j} = \frac{a_{n-1,1}a_{1j}}{a_{11}} \quad (1 \leq j \leq n-1).$$

Set  $a_{1n} := c_{1n}$ . By  $c_{1n}(c_{2w(2)} \cdots c_{n-1w(n-1)})c_{n1} = 1$  for  $w \in A_n$  such that  $w(1) = n$ ,  $w(n) = 1$ ,

$$\begin{aligned} a_{1n} \cdot \frac{a_{12} \cdots a_{1,n-1}a_{21} \cdots a_{n-1,1}}{a_{11}^{n-2}} \cdot c_{n1} &= 1 \\ \therefore c_{n1} &= \frac{a_{11}^{n-2}}{a_{12} \cdots a_{1n}a_{21} \cdots a_{n-1,1}} = \frac{a_{11}z}{a_{1n}a_{n-1,1}}. \end{aligned}$$

Set  $a_{n1} := c_{n1}$ . Then

$$z = \frac{a_{1n}a_{n-1,1}a_{n1}}{a_{11}}.$$

It follows that  $c_{ij} = \frac{a_{i1}a_{1j}}{a_{11}}$  for  $1 \leq i, j \leq n-1$  and  $(i, j) = (1, n), (n, 1), (n, n)$  where  $a_{11} \cdots a_{1n}a_{11} \cdots a_{n1} = a_{11}^n$ . By  $c_{in}(c_{1w(1)} \cdots \widehat{c_{in}} \cdots c_{n-1w(n-1)})c_{n1} = 1$ , we have

$$\begin{aligned} c_{in} \cdot \frac{a_{12} \cdots a_{1,n-1}a_{21} \cdots a_{n-1,1}}{a_{11}^{n-3}a_{i1}} \cdot a_{n1} &= 1 \\ \therefore c_{in} &= \frac{a_{11}^{n-3}a_{i1}}{a_{12} \cdots a_{1,n-1}a_{21} \cdots a_{n1}} = \frac{a_{i1}a_{1n}}{a_{11}}. \end{aligned}$$

By  $(c_{1w(1)} \cdots c_{n-1w(n-1)})c_{nj} = 1$ , we have

$$\begin{aligned} \frac{a_{12} \cdots a_{1n}a_{11} \cdots a_{n-1,1}}{a_{11}^{n-1}a_{1j}} \cdot c_{nj} &= 1 \\ \therefore c_{nj} &= \frac{a_{11}^{n-1}a_{1j}}{a_{12} \cdots a_{1n}a_{11} \cdots a_{n-1,1}} = \frac{a_{n1}a_{1j}}{a_{11}}. \end{aligned}$$

From the above, we obtain (7).

Each of  $2 \times 2$  minor determinants of  $C$  is 0, so that  $\text{rank}(C) = 1$  follows.  $\square$

*Proof of Theorem 3.* Let  $T^{-1} \in \text{Stab}(\det_n^{(\alpha, \beta)})$  and  $\alpha \neq \pm\beta$ . By Lemma 6 and 7, We can write  $T(X) = C * PXQ$  or  $T(X) = C * P^tXQ$  where  $\text{sgn}(\sigma)\text{sgn}(\tau) = 1$ . By Lemma 8, we can write  $c_{ij} = l_i r_j$  ( $1 \leq i, j \leq n$ ) where  $l_1 \cdots l_n r_1 \cdots r_n = 1$ . Set  $L := \text{diag}(l_1, \dots, l_n)$  and  $R := \text{diag}(r_1, \dots, r_n)$ . Then we obtain  $T(X) = LPXQR$  or  $T(X) = LP^tXQR$  where  $\text{sgn}(\sigma)\text{sgn}(\tau) = 1$ , which proves the theorem.  $\square$

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